Известия Гомельского государственного университета
имени Ф. Скорины, № 3 (144), 2024

Физика

УДК 539.12 EDN: UDKGGS

Internal symmetry of the quantized multi-component Majorana fields

P.P. ANDRUSEVICH[1], V.M. RED'KOV[2]

We continue the investigation of the internal symmetry of the multi-component Majorana fields, which can be associated with one, two, three and four Dirac fields. The case of the classical fields was studied in the previous paper. According to the definition, internal symmetries should satisfy a number of requirements: the corresponding transformations must preserve the form of the equation which leads to the constrains $[Q,\Gamma_\mu]_- = 0$ in a massive case, and to $[Q,\Gamma_\mu]_- = 0$ and $[Q,\Gamma_\mu]_+ = 0$ for massless fields; the Lagrangian must be invariant under such symmetries, which yields to the constraint $Q^+\eta Q = \eta$; the transformations must preserve the Majorana nature of the fields, that means that if Ψ_A is the real (imaginary) part of the wave function, then after the transformation $\Psi'_A = Q_{AB}\Psi_B$ it remains real (imaginary). For the quantized fields, the following permutation relations are added as additional constraints $[\psi_i,\bar{\psi}_j]_+ = (\gamma_4)_{ij}$, $[\bar{\psi}_i,\psi_j]_+ = -(\gamma_4)_{ij}$, $[\psi_i,\psi_j]_+ = 0$, $\bar{\psi}_i,[\bar{\psi}_j]_+ = 0$. We describe the structure of symmetry transformations for the cases of 1, 2, 3, and 4 quantized Dirac fields, massive and massless, specifying them to Majorana case.
Keywords: quantized dirac fields, majorana fields, lagrangian formalism, internal symmetry.

Мы продолжаем исследование внутренней симметрии многокомпонентных полей Майораны, которые могут быть связаны с одним, двумя, тремя и четырьмя полями Дирака. Случай классических полей был изучен в предыдущей работе. Согласно определению, внутренние симметрии должны удовлетворять ряду требований: соответствующие преобразования должны сохранять форму уравнения, это приводит к ограничениям $[Q,\Gamma_\mu]_- = 0$ в массовом случае, а также к $[Q,\Gamma_\mu]_- = 0$ и $[Q,\Gamma_\mu]_+ = 0$ для безмассовых полей; лагранжиан должен быть инвариантным, что приводит к ограничению $Q^+\eta Q = \eta$; преобразования не должны нарушать майорановский характер полей, т. е. эти преобразования должны удовлетворять условию: если Ψ_A – вещественная (мнимая) часть волновой функции, то и $\Psi'_A = Q_{AB}\Psi_B$ – также вещественная (мнимая) часть. Для квантованных полей в качестве дополнительных ограничений добавляются следующие перестановочные соотношения $[\psi_i,\bar{\psi}_j]_+ = (\gamma_4)_{ij}$, $[\bar{\psi}_i,\psi_j]_+ = -(\gamma_4)_{ij}$, $[\psi_i,\psi_j]_+ = 0$, $\bar{\psi}_i,[\bar{\psi}_j]_+ = 0$. Мы описываем структуру преобразований симметрии для случаев 1, 2, 3 и 4 квантованных полей Дирака, массивных и безмассовых, уточнив их для случая Майораны.
Ключевые слова: квантованные дираковские поля, майорановские поля, лагранжев формализм, внутренняя симметрия.

Introduction. We continue the investigation of the internal symmetry of the multi-component Majorana fields, which can be associated with one, two, three and four Dirac fields (see in [1], [2]). The case of the classical fields was studied in the previous paper (see [1], [2] and the list of reference therein).

In the quantized theory for the Dirac fields, in addition to bispinor variable ψ they introduce Dirac conjugate bispinor $\bar{\psi} = \psi^+\gamma_4$. The anticommutators for operator variables ψ and ψ^+ take the form [3]

$$[\psi'_i,\psi'^+_j]_+ = \delta_{ij}, [\psi'_i,\psi'_j]_+ = 0, [\psi'^+_i,\psi'^+_j]_+ = 0. \qquad (1)$$

Anticommutators. In these formulas the Dirac matrices γ'_μ are taken as follows

$$\gamma_1' = \begin{vmatrix} & & & -i \\ & & -i & \\ & i & & \\ i & & & \end{vmatrix}, \ \gamma_2' = \begin{vmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{vmatrix}, \ \gamma_3' = \begin{vmatrix} & & -i & \\ & & & i \\ i & & & \\ & -i & & \end{vmatrix}, \ \gamma_4' = \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{vmatrix},$$

which can be translated to Majorana form by means of the similarity transformation

$$\gamma_\mu = S\gamma_\mu' S^{-1}, S = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & -i & i & 0 \end{vmatrix}, \ S^{-1} = S^+ = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & i \\ -1 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{vmatrix}. \quad (2)$$

Let us transform anticommutators (1) to Majorana basis. Starting with definitions

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4), \quad \psi' = (\psi_1', \psi_2', \psi_3', \psi_4'),$$

we get $\psi_1 = -\psi_2' - \psi_3'$, $\psi_2 = \psi_1' - \psi_4'$, $\psi_3 = -i\psi_1' - i\psi_4'$, $\psi_4 = -i\psi_2' + i\psi_3'$, further we verify that the above relations (1) preserve their form for the new components ψ:

$$[\psi_i, \psi_j^+]_+ = \delta_{ij}, \quad [\psi_i, \psi_j]_+ = [\psi_i^+, \psi_j^+]_+ = 0. \quad (3)$$

Now let us introduce the Dirac conjugate variables in Majorana basis

$$\bar{\psi} = \psi^+\gamma_4 = i(\psi_4^+, -\psi_3^+, \psi_2^+, -\psi_1^+),$$

and find the commutation relations for the variables $\psi, \bar{\psi}$:

$$[\psi_i, \bar{\psi}_j]_+ = (\gamma_4)_{ij}, \quad [\bar{\psi}_i, \psi_j]_+ = -(\gamma_4)_{ij}, \quad [\psi_i, \psi_j]_+ = 0, \quad [\bar{\psi}_i, \bar{\psi}_j]_+ = 0. \quad (4)$$

It is evident that these relations are invariant under the choice of basis in bispinor space.

Technical details. Let us elaborate a special method to work with multicomponent columns. Let A and B be some matrices of dimension $n \times 1$. It is convenient to introduce a special operation over such matrices $\{A, B^T\}_\oplus = A \otimes B^T + B^T \otimes A$, where symbol T designates the transposition of the matrix. This operation has several helpful properties:

$$\{A, (B_1 + B_2)^T\}_\oplus = \{A, B_1^T\}_\oplus + \{A, B_2^T\}_\oplus, \quad (5)$$

$$(\{A, B^T\}_\oplus)^T = (A \otimes B^T + B^T \otimes A)^T = A^T \otimes B + B \otimes A^T = \{B, A^T\}_\oplus. \quad (6)$$

Let x be a certain matrix of dimension $n \times n$, then

$$\{A, (xB)^T\}_\oplus = \{A, B^T\}_\oplus x^T. \quad (7)$$

Let the matrices A and B contain two blocks

$$A = \begin{vmatrix} a_1 \\ a_2 \end{vmatrix}, \quad B^T = \begin{vmatrix} b_1^T & b_2^T \end{vmatrix},$$

then in accordance with the definitions we derive the identity

$$\left\{ \begin{vmatrix} a_1 \\ a_2 \end{vmatrix}, \ \begin{vmatrix} b_1^T & b_2^T \end{vmatrix} \right\}_\oplus = \begin{vmatrix} \{a_1, b_1^T\}_\oplus & \{a_1, b_2^T\}_\oplus \\ \{a_2, b_1^T\}_\oplus & \{a_2, b_2^T\}_\oplus \end{vmatrix}. \quad (8)$$

One Dirac Quantized Field. Let us consider one Dirac quantized field in Majorana basis

$$(\Gamma_\mu \partial_\mu + m)\Psi' = 0, \ \Gamma_\mu = I_2 \otimes \gamma_\mu, \Psi' = (\psi^r, \psi^i). \quad (9)$$

Further we will use the 8-component variable (adopted to quantum field theory) $\Psi = (\psi, \bar{\psi})$. The transition from Ψ to Ψ' is reached by means of the unitary transformation

$$U = \frac{1}{\sqrt{2}}\begin{vmatrix} I_4 & I_4 \\ \gamma_2 & -\gamma_4 \end{vmatrix}, \ U^{-1} = \frac{1}{\sqrt{2}}\begin{vmatrix} I_4 & \gamma_4 \\ I_4 & -\gamma_4 \end{vmatrix}, \ \Psi = U\Psi' = \begin{vmatrix} \psi \\ \gamma_4\psi^* \end{vmatrix} = \begin{vmatrix} \psi \\ \bar{\psi} \end{vmatrix}. \quad (10)$$

We can identify the variables Ψ, Ψ^T with block variables in the above quantities A, B^T. Then we get

$$\{\Psi, \Psi^T\}_\oplus = \left\{ \begin{vmatrix} \psi \\ \bar{\psi} \end{vmatrix}, \ \begin{vmatrix} \psi^T & \bar{\psi}^T \end{vmatrix} \right\}_\oplus = \begin{vmatrix} \{\psi, \psi^T\}_\oplus & \{\psi, \bar{\psi}^T\}_\oplus \\ \{\bar{\psi}, \psi^T\}_\oplus & \{\bar{\psi}, \bar{\psi}^T\}_\oplus \end{vmatrix}. \quad (11)$$

According to previous Section, the anticommutation relations (4) hold. Let us specify them for the quantities from (11). For $\{\psi,\bar{\psi}^T\}_\oplus$, taking in mind (1), we get $\{\psi,\bar{\psi}^T\}_\oplus=\gamma_4$. Similarly, we find

$$\{\psi,\psi^T\}_\oplus=\{\bar{\psi},\quad \bar{\psi}^T\}_\oplus=0,\quad \{\bar{\psi},\psi^T\}_\oplus=(\{\psi,\bar{\psi}^T\}_\oplus)^T=(\gamma_4)^T=-\gamma_4.$$

These four relations can be presented in the shorter form as follows

$$\{\Psi,\Psi^T\}_\oplus=\begin{vmatrix}0 & \gamma_4\\ -\gamma_4 & 0\end{vmatrix} \Rightarrow \{\Psi,\Psi^T\}_\oplus=i\sigma_2\otimes\gamma_4\,; \tag{12}$$

it contains four anticommutators in 8-dimensional form.

This formalism can be extended to any number of the Dirac fields. Let n Dirac fields be given

$$(\gamma_\mu\partial_\mu+m)\psi_1=0,\quad (\gamma_\mu\partial_\mu+m)\psi_2=0,\quad (\gamma_\mu\partial_\mu+m)\psi_n=0. \tag{13}$$

For system (13) in Majorana basis, we have $2n$-component variable Ψ with the (column) structure $\Psi=(\psi^r,\psi^i)$, where the column $\psi^r=(\psi_1^r,\psi_2^r,\ldots,\psi_n^r)$ is real, and the column $\psi^i=(\psi_1^i,\psi_2^i,\ldots,\psi_n^i)$ is imaginary. The $2n$-component function takes the form $\Psi=(\psi,\bar{\psi})$ where $\psi=(\psi_1,\psi_2,\ldots,\psi_n)$, $\bar{\psi}=(\bar{\psi}_1,\bar{\psi}_2,\ldots,\bar{\psi}_n)$. For this generalized case, the commutative relations are written as follows (the indices α,β numerate n fields)

$$[\psi_{i\alpha},\bar{\psi}_{j\beta}]_+=\delta_{\alpha\beta}(\gamma_4)_{ij},\ [\bar{\psi}_{i\alpha},\psi_{j\beta}]_+=-\delta_{\alpha\beta}(\gamma_4)_{ij},\ [\psi_{i\alpha},\psi_{j\beta}]_+=[\bar{\psi}_{i\alpha},\bar{\psi}_{j\beta}]_+=0. \tag{14}$$

By analogy with the case of one Dirac equation, we derive

$$\{\psi,\bar{\psi}^T\}_\oplus=I_n\otimes\gamma_4,\ \{\bar{\psi},\psi^T\}_\oplus=-I_n\otimes\gamma_4,\ \{\psi,\psi^T\}_\oplus=0,\ \{\bar{\psi},\bar{\psi}^T\}_\oplus=0. \tag{15}$$

These four relations can be joint into the following one

$$\{\Psi,\Psi^T\}_\oplus=i\sigma_2\otimes(I_n\otimes\gamma_2). \tag{16}$$

Symmetries for n Dirac field. Let us consider infinitesimal 1-parametric transformation over the field

$$\Psi'=(1+\omega J)\Psi, \tag{17}$$

where J is any generator of internal symmetry. Let us examine the behavior of the anti-commutative relations under this transformation

We start with the relation

$$\{\Psi',\Psi'^T\}_\oplus=(1+\omega J)\Psi\otimes\Psi^T(1+\omega J^T)+\Psi^T(1+\omega J^T)\otimes(1+\omega J)\Psi\,; \tag{18}$$

after transforming the first and the second summands, preserving the terms of the first order in ω:

$$(1+\omega J)\Psi\otimes\Psi^T(1+\omega J^T)=\Psi\otimes\Psi^T+\omega\Psi\otimes\Psi^T J^T+\omega J\Psi\otimes\Psi^T,$$

$$\Psi^T(1+\omega J^T)\otimes(1+\omega J)\Psi=\Psi^T\otimes\Psi+\omega\Psi^T\otimes J\Psi+\omega\Psi^T J^T\otimes\Psi,$$

whence for (18), we obtain

$$\{\Psi',\Psi'^T\}_\oplus=\{\Psi,\Psi^T\}_\oplus+\omega\{\Psi,(J\Psi)^T\}_\oplus+\omega\{\Psi^T,(J\Psi)\}_\oplus \tag{19}$$

We should impose the constraint on symmetry generator J $\omega\{\Psi,(J\Psi)^T\}_\oplus+\omega\{\Psi^T,(J\Psi)\}_\oplus=0$, whence taking into account (6) we derive

$$\{\Psi,(J\Psi)^T\}_\oplus=-(\{\Psi,(J\Psi)^T\}_\oplus)^T, \tag{20}$$

in the following we will call the last as the anti-symmetry condition.

Let us examine the consequences of (20). To this end, let us use the block form of internal symmetry generator in the basis $(\psi,\bar{\psi})$

$$J=\begin{vmatrix}a & b\\ c & d\end{vmatrix}, \tag{21}$$

where

$$a=\begin{vmatrix}a_{11} & \ldots & a_{1n}\\ \vdots & & \vdots\\ a_{n1} & \ldots & a_{nn}\end{vmatrix},\ b=\begin{vmatrix}b_{11} & \ldots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \ldots & b_{nn}\end{vmatrix},\ c=\begin{vmatrix}c_{11} & \ldots & c_{1n}\\ \vdots & & \vdots\\ c_{n1} & \ldots & c_{nn}\end{vmatrix},\ d=\begin{vmatrix}d_{11} & \ldots & d_{1n}\\ \vdots & & \vdots\\ d_{n1} & \ldots & d_{nn}\end{vmatrix},$$

the blocks $a_{ij},b_{ij},c_{ij},d_{ij}$ have dimension 4×4. Let us calculate the quantities

$$J\Psi = \begin{vmatrix} a\psi + b\overline{\psi} \\ c\psi + d\overline{\psi} \end{vmatrix}, (J\Psi)^T = \left| (a\psi)^T + (b\overline{\psi})^T, \quad (c\psi)^T + (d\overline{\psi})^T \right|,$$

$$\{\Psi, (J\Psi)^T\}_\oplus = \left\{ \begin{vmatrix} \psi \\ \overline{\psi} \end{vmatrix}, \left| (a\psi)^T + (b\overline{\psi})^T, \quad (c\psi)^T + (d\overline{\psi})^T \right| \right\}_\oplus .$$

First, let us detail the term $\{\psi, (b\overline{\psi})^T\}_\oplus$: $\{\psi, (b\overline{\psi})^T\}_\oplus = \psi \otimes (b\overline{\psi})^T + (b\overline{\psi})^T \otimes \psi$; whence taking in mind the identity

$$b\overline{\psi} = \begin{vmatrix} \Sigma b_{1i}\overline{\psi}_i \\ \vdots \\ \Sigma b_{ni}\overline{\psi}_i \end{vmatrix}, (b\overline{\psi})^T = \left| (\Sigma b_{1i}\overline{\psi}_i)^T \quad \ldots \quad (\Sigma b_{ni}\overline{\psi}_i)^T \right|$$

we obtain

$$\{\psi, (b\overline{\psi})^T\}_\oplus = \left\{ \begin{vmatrix} \psi_1 \\ \vdots \\ \psi_n \end{vmatrix}, \left| (\Sigma b_{1i}\overline{\psi}_i)^T \quad \ldots \quad (\Sigma b_{ni}\overline{\psi}_i)^T \right| \right\}_\oplus = \begin{vmatrix} \{\psi_1, (\Sigma b_{1i}\overline{\psi}_i)^T\}_\oplus & \ldots & \{\psi_1, (\Sigma b_{ni}\overline{\psi}_i)^T\}_\oplus \\ \vdots & & \vdots \\ \{\psi_n, (\Sigma b_{1i}\overline{\psi}_i)^T\}_\oplus & \ldots & \{\psi_n, (\Sigma b_{ni}\overline{\psi}_i)^T\}_\oplus \end{vmatrix}.$$

Now let us detail the term

$$\{\psi_1, (\Sigma b_{1i}\overline{\psi}_i)^T\}_\oplus = \{\psi_1, (b_{11}\overline{\psi}_1)^T\}_\oplus + \{\psi_1, (b_{12}\overline{\psi}_2)^T\}_\oplus + \ldots + \{\psi_1, (b_{1n}\overline{\psi}_n)^T\}_\oplus, \tag{22}$$

by direct calculation we can prove that for each bispinor the identity holds (where x is certain matrix of dimension 4×4)

$$\{\psi, (x\overline{\psi})^T\}_\oplus = \{\psi, \overline{\psi}^T\}_\oplus x^T .$$

Taking into account the last identity and the property from (22) we derive

$$\{\psi, (b\overline{\psi})^T\}_\oplus = \begin{vmatrix} \{\psi_1, \overline{\psi}_1\}_\oplus b_{11}^T & \ldots & \{\psi_1, \overline{\psi}_1\}_\oplus b_{n1}^T \\ \vdots & & \vdots \\ \{\psi_n, \overline{\psi}_n\}_\oplus b_{1n}^T & \ldots & \{\psi_n, \overline{\psi}_n\}_\oplus b_{nn}^T \end{vmatrix} = \begin{vmatrix} \{\psi_1, \overline{\psi}_1\}_+ & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & \{\psi_n, \overline{\psi}_n\}_+ \end{vmatrix} \begin{vmatrix} b_{11} & \ldots & b_{n1} \\ \vdots & & \vdots \\ b_{1n} & \ldots & b_{nn} \end{vmatrix} = \{\psi, \overline{\psi}\}_\oplus b^T .$$

Similarly, we can derive the identities (where symbol y designates blocks from (21)):

$$\{\psi, (y\overline{\psi})^T\}_\oplus = -\{\overline{\psi}, (y\psi)^T\}_\oplus = \{\psi, \overline{\psi}^T\}_\oplus y^T, \quad \{\psi, (y\psi)^T\}_\oplus = \{\overline{\psi}, (y\overline{\psi})^T\}_\oplus = 0, \tag{23}$$

Therefore, expression $\{\Psi, (J\Psi)^T\}_\oplus$ is transformed to

$$\{\Psi, (J\Psi)^T\}_\oplus = \begin{vmatrix} 0 & \{\psi, \overline{\psi}^T\}_\oplus \\ -\{\psi, \overline{\psi}^T\}_\oplus & 0 \end{vmatrix} \begin{vmatrix} a^T & c^T \\ b^T & d^T \end{vmatrix} = \{\Psi, \Psi^T\}_\oplus J^T . \tag{24}$$

Whence we derive yet another identity $(\{\Psi, (J\Psi)^T\}_\oplus)^T = (\{\Psi, \Psi^T\}_\oplus J^T)^T = J(\{\Psi, \Psi^T\}_\oplus)^T$.

Thus, the symmetry requirement for commutative relations (20) can be presented in the form

$$\{\Psi, \Psi^T\}_\oplus J^T = -J(\{\Psi, \Psi^T\}_\oplus)^T , \tag{25}$$

which with (16) in mind leads to the restriction on symmetry generator

$$i\sigma_2 \otimes (I_n \otimes \gamma_4) J^T = -Ji\sigma_2 \otimes (I_n \otimes \gamma_4). \tag{26}$$

It should be noted that having used the property (7), one can present expression (20) in the form (25).

Let us recall that in the basis (ψ^r, ψ^i) for system (13), the general structure of the symmetry generators should be as follows

$$J = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \otimes I_4 , \tag{27}$$

where a, b, c, d stand for the blocks of dimension $n \times n$ (they differ from blocks in (21)). Let us translate(27) to the basis $(\psi, \overline{\psi})$, $\overline{J} = UJU^{-1}$, with the help of the matrix

$$U = \frac{1}{\sqrt{2}} \begin{vmatrix} I_n \otimes I_4 & I_n \otimes I_4 \\ I_n \otimes \gamma_4 & -I_n \otimes \gamma_4 \end{vmatrix}, \quad U^{-1} = U^+ = \frac{1}{\sqrt{2}} \begin{vmatrix} I_n \otimes I_4 & I_n \otimes \gamma_4 \\ I_n \otimes I_4 & -I_n \otimes \gamma_4 \end{vmatrix}. \tag{28}$$

Taking in mind the identities

$$UJ = \begin{vmatrix} I_n \otimes I_4 & I_n \otimes I_4 \\ I_n \otimes \gamma_4 & -I_n \otimes \gamma_4 \end{vmatrix} \begin{vmatrix} a \otimes I_4 & b \otimes I_4 \\ c \otimes I_4 & d \otimes I_4 \end{vmatrix} = \begin{vmatrix} (a+c) \otimes I_4 & (b+d) \otimes I_4 \\ (a-c) \otimes \gamma_4 & (b-d) \otimes \gamma_4 \end{vmatrix},$$

$$UJU^{-1} = \begin{vmatrix} (a+c) \otimes I_4 & (b+d) \otimes I_4 \\ (a-c) \otimes \gamma_4 & (b-d) \otimes \gamma_4 \end{vmatrix} \times \begin{vmatrix} I_n \otimes I_4 & I_n \otimes \gamma_4 \\ I_n \otimes I_4 & -I_n \otimes \gamma_4 \end{vmatrix},$$

we find the structure of the generator $\bar{J}$:

$$\bar{J} = \begin{vmatrix} \alpha \otimes I_4 & \beta \otimes \gamma_4 \\ \sigma \otimes \gamma_4 & \rho \otimes I_4 \end{vmatrix}, \tag{29}$$

where the notations are used

$$\alpha = a+b+c+d, \ \beta = a+c-b-d, \ \sigma = a-c+b-d, \ \rho = a-c-b+d. \tag{30}$$

We substitute this generator (29) into (26); in this way with the use of identities

$$i\sigma_2 \otimes (I_n \otimes \gamma_4)\bar{J}^T = \begin{vmatrix} 0 & I_n \otimes \gamma_4 \\ -I_n \otimes \gamma_4 & 0 \end{vmatrix} \begin{vmatrix} \alpha^T \otimes I_4 & -\sigma^T \otimes \gamma_4 \\ -\beta^T \otimes \gamma_4 & \rho^T \otimes I_4 \end{vmatrix} = \begin{vmatrix} -\beta^T \otimes I_4 & \rho^T \otimes \gamma_4 \\ -\alpha^T \otimes \gamma_4 & \sigma^T \otimes I_4 \end{vmatrix},$$

$$\bar{J} i\sigma_2 \otimes (I_n \otimes \gamma_4) = \begin{vmatrix} \alpha \otimes I_4 & \beta \otimes \gamma_4 \\ \sigma \otimes \gamma_4 & \rho \otimes I_4 \end{vmatrix} \begin{vmatrix} 0 & I_n \otimes \gamma_4 \\ -I_n \otimes \gamma_4 & 0 \end{vmatrix} = \begin{vmatrix} -\beta \otimes I_4 & \alpha \otimes \gamma_4 \\ -\rho \otimes \gamma_4 & \sigma \otimes I_4 \end{vmatrix},$$

we derive

$$\begin{vmatrix} -\beta \otimes I_4 & \alpha \otimes \gamma_4 \\ -\rho \otimes \gamma_4 & \sigma \otimes I_4 \end{vmatrix} = -\begin{vmatrix} -\beta^T \otimes I_4 & \rho^T \otimes \gamma_4 \\ -\alpha^T \otimes \gamma_4 & \sigma \otimes I_4 \end{vmatrix}.$$

Whence we obtain the following restrictions on the blocks $\alpha, \beta, \rho, \sigma$:

$$\beta = -\beta^T, \ \alpha = -\rho^T, \ \rho = -\alpha^T, \ \sigma = -\sigma^T. \tag{31}$$

Let us substitute the expression from (30) into (31):

$$a+c-b-d = -a^T - c^T + b^T + d^T, \quad a-c+b-d = -a^T + c^T - b^T + d^T,$$

$$a-c-b+d = -a^T - c^T - b^T - d^T, \quad a+c+b+d = -a^T + c^T + b^T - d^T,$$

whence it follows

$$a-d = -a^T + d^T, \ c-b = -c^T + b^T, \ a+d = -a^T - d^T, \ -c-b = -c^T - b^T,$$

that is

$$a = -a^T, \quad d = -d^T, \quad c = b^T. \tag{32}$$

Therefore, the structure of the symmetry generator in the basis (ψ^r, ψ^i) should be as follows

$$J = \begin{vmatrix} a & \\ & d \end{vmatrix} \otimes I_4 + \begin{vmatrix} & b \\ b^T & \end{vmatrix} \otimes I_4. \tag{33}$$

With this in mind, for generator $\bar{J}$ we obtain

$$\bar{J} = \begin{vmatrix} \alpha \otimes I_4 & \beta \otimes \gamma_4 \\ \sigma \otimes \gamma_4 & -\alpha^T \otimes I_4 \end{vmatrix}. \tag{34}$$

Thus, we have studied internal symmetries for 1, 2, 3, and 4 Dirac quantized fields. The needed generators in the basis (ψ^r, ψ^i) may have the structure of two types:

$$\omega_D J_D \Psi = \omega_D \begin{vmatrix} D_1 & \\ & D_2 \end{vmatrix} \begin{vmatrix} \psi^r \\ \psi^i \end{vmatrix}, \quad \omega_A J_A \Psi = \omega_A \begin{vmatrix} & A_1 \\ A_2 & \end{vmatrix} \begin{vmatrix} \psi^r \\ \psi^i \end{vmatrix}, \tag{35}$$

where the blocks D_k are real, and A_k are imaginary; the parameters ω_D and ω_A are imaginary ($k = 1, 2$). All the generators are Hermitian, which yields

$$D_k^+ = ((D_k)^T)^* = -D_k^T = D_k, \quad A_k^+ = ((A_k)^T)^* = A_k^T = A_k, \quad l = 1, 2. \tag{36}$$

All the transformations with such properties preserve the form of the commutation relations (4). Expressions (35) and (36) agree with the needed symmetry conditions for classical field.

Thus, we conclude that all symmetry transformations for 1, 2, 3, 4 massive field preserve their validness in a quantized case.

Massless quantized fields. Let us examine invariance properties of the commutation relations for massless fields. In addition to generators of type (27) in the basis (ψ^r, ψ^i), in massless case we have generators with different structure (compare with (27)):

$$L = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \otimes \gamma_5, \tag{37}$$

where a, b, c, d stand for the blocks of dimension $n \times n$ (they differ from the blocks (21)).

Let us transform relations (37) to basis $(\psi, \overline{\psi})$, with the help of matrix (28): $\overline{L} = ULU^{-1}$. Taking into account the identities

$$UL = \begin{vmatrix} I_n \otimes I_4 & I_n \otimes I_4 \\ I_n \otimes \gamma_4 & -I_n \otimes \gamma_4 \end{vmatrix} \begin{vmatrix} a \otimes \gamma_5 & b \otimes \gamma_5 \\ c \otimes \gamma_5 & d \otimes \gamma_5 \end{vmatrix} = \begin{vmatrix} (a+c) \otimes \gamma_5 & (b+d) \otimes \gamma_5 \\ (a-c) \otimes \gamma_4\gamma_5 & (b-d) \otimes \gamma_4\gamma_5 \end{vmatrix},$$

$$ULU^{-1} = \begin{vmatrix} (a+c) \otimes \gamma_5 & (b+d) \otimes \gamma_5 \\ (a-c) \otimes \gamma_4\gamma_5 & (b-d) \otimes \gamma_4\gamma_5 \end{vmatrix} \begin{vmatrix} I_n \otimes I_4 & I_n \otimes \gamma_4 \\ I_n \otimes I_4 & -I_n \otimes \gamma_4 \end{vmatrix},$$

we find the structure of generators $\overline{L}$:

$$\overline{L} = \begin{vmatrix} \alpha \otimes \gamma_5 & \beta \otimes \gamma_4\gamma_5 \\ \sigma \otimes \gamma_4\gamma_5 & \rho \otimes \gamma_5 \end{vmatrix}, \tag{38}$$

where the notations are used

$$\alpha = a+b+c+d, \;\; \beta = -a-c+b+d, \;\; \sigma = a-c+b-d, \;\; \rho = -a+c+b-d. \tag{39}$$

Substituting (38) – (39) into (26), we get

$$\begin{vmatrix} -\beta \otimes I_4 & \alpha \otimes \gamma_4 \\ -\rho \otimes \gamma_4 & \sigma \otimes I_4 \end{vmatrix} = -\begin{vmatrix} -\beta^T \otimes I_4 & \rho^T \otimes \gamma_4 \\ -\alpha^T \otimes \gamma_4 & \sigma \otimes I_4 \end{vmatrix},$$

whence follow the restrictions on the blocks $\alpha, \beta, \rho, \sigma$:

$$\beta = \beta^T, \alpha = -\rho^T, \quad \rho = -\alpha^T, \sigma = \sigma^T. \tag{40}$$

Substituting (40) into (39), we obtain the system of equations

$$-a-c+b+d = -a^T - c^T + b^T + d^T, \quad -a+c+b-d = -a^T - c^T - b^T - d^T,$$

$$-a-c-b-d = -a^T + c^T + b^T - d^T, \quad +a-c+b-d = +a^T - c^T + b^T - d^T,$$

whence it follows

$$a = a^T, \quad d = d^T, \quad c = -b^T. \tag{41}$$

Thus, the structure of the generators for symmetry transformations in the basis (ψ^r, ψ^i) is given by the formula

$$L = \begin{vmatrix} a & \\ & d \end{vmatrix} \otimes \gamma_5 + \begin{vmatrix} & b \\ -b^T & \end{vmatrix} \otimes \gamma_5, \tag{42}$$

recall that a, d obey the constraints (42).

Previously it was shown that all appropriate generators of type L in the basis (ψ^r, ψ^i) may be of two types:

$$D, \quad L^d = \begin{vmatrix} d_1 & \\ & d_2 \end{vmatrix} \otimes \gamma_5 \quad A, \quad L^a = \begin{vmatrix} & a_1 \\ a_2 & \end{vmatrix} \otimes \gamma_5.$$

The Majorana condition assumes that generators of type D are to have imaginary blocks *m* and the generators of type A are to have real blocks. Because all generators are Hermitian, we have the properties

$$d^+ = ((d)^T)^* = -d^T = d, \quad a^+ = ((a)^T)^* = a^T = a.$$

The two last relation contradict to (42), for this reason we conclude that for quantized massless fields only the generators of type J provide us with symmetry transformation, exactly as in massive case.

Conclusion. In the present paper, the problem of describing the internal symmetry transformations for quantized Dirac fields has been studied. We started with the matrix equation $(\Gamma_\mu \partial_\mu + m)\psi = 0$, and introduced the concept of the internal symmetry. These symmetries should preserve the form of the basic equation, which is equivalent to the commutation relation $[Q, \Gamma_\mu]_- = 0$. The relevant Lagrangian should be invariant under the internal symmetry transformation. This requirement leads to the restriction $Q^+ \eta Q = \eta$, where η stands for the bilinear form matrix. We impose one additional requirement on symmetry transformations, such transformations should preserve the Majorana nature of the fields.

The situation for massless case is substantially different, $\Gamma_\mu \partial_\mu \psi = 0$. The requirement of invariance of this equation and corresponding Lagrangian leads to two alternative restrictions $[Q_1, \Gamma_\mu]_- = 0$ or $[Q_2, \Gamma_\mu]_+ = 0$. The Lagrangian invariance with respect to internal symmetry transformation for massless case coincide with that for massive case, $Q^+ \eta Q = \eta$.

It is proved that in massive and massless cases the internal symmetry transformations are determined by the same generators, which coincide with the generators established when studying the classical fields.

The authors are grateful to Professor V.A. Pletyuchov for encouraging help and advice.

References

1. Andrusevich, P. P. Dirac like equations and generalized Majorana fields, symmetry aspects / P. P. Andrusevich // Nonlinear Dynamics and Applications : proceedings of the twenty eight Anniversary Seminar NPCS-2022, Minsk, June 18–21, 2022 = Нелинейная динамика и приложения : труды XXVIII Международного семинара, Минск, 21–21 июня 2022 г. / редкол.: В. А. Шапоров [и др.] ; под ред. В. А. Шапорова, А. Г. Трифонова ; Объединенный институт энергетических и ядерных исследований – «Сосны» НАН Беларуси. – Минск : Право и экономика, 2022. – 486 с.

2. Andrusevich, P. P. Dirac like equations and generalized Majorana fields, internal symmetry / P. P. Andrusevich, V. M. Red'kov // Problems of Physics, Mathematics and Techniques. – 2024. – P. 7–15.

3. Kushnirenko, A. N. Introduction to quantum field theory / A. N. Kushnirenko. – Moscow, 1971. – 304 p.

[1]Брестский государственный
колледж связи

[2]Институт физики имени Б.И. Степанова
НАН Беларуси

Поступила в редакцию 20.12.2023